

Meadow based Fraction Theory

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Abstract

In the context of an involutive meadow a precise definition of fractions is formulated and on that basis formal definitions of various classes of fractions are given. The definitions follow the fractions as terms paradigm. That paradigm is compared with two competing paradigms for storytelling on fractions: fractions as values and fractions as pairs.¹

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1 Introduction

The notion of a fraction is a difficult one.² I will first discuss three positions on fractions defended respectively by Pierre van Hiele, Friedhelm Padberg, and Stefan Rollnik. Van Hiele in [22] suggests to do away with fractions entirely and only to use the inverse function instead. Van Hiele considered fractions to be a topic that leads to formidable problems throughout teaching and he expected that thinking about inverses instead of divisions will make matters more accessible. There is no indication that this 50 year old proposal gained much or even any support. The advantages claimed by van Hiele are not easy to appreciate.

Padberg [27] assumes that fraction is a complex notion amenable to a thematic decomposition. Decomposing the notion of a fractions in so-called subconstructs originates from Kieren [24]. Different subconstructs go with a different langue and notation and even a different way of thinking, thus constituting different logics of fractions so to speak. The necessity of this conceptual complexity seems to be unproven, however.

1.1 Fractions as values paradigm

Rollnik in [30] provides a proposal for thinking about fractions based on the view that a fraction is a number, in particular a rational number. In the setting

²In [33] one finds a comprehensive survey of the history and development of fractions.

of the current paper Rollnik’s view corresponds to a fraction merely being an element of a particular involutive meadow, the meadow \mathbb{Q}_0 of rational numbers. I propose to label the position that a fraction stands for a number as the fraction as a value view, or alternatively as the fractions as values paradigm. Below this paradigm will be contrasted with two other paradigms on fractions.³

A remarkable consequence of the fractions as values paradigm view is that upon its adoption it becomes meaningless to speak of the numerator and of the denominator of a fraction.⁴

A further price paid when adopting the fractions as values paradigm is that names of numbers, and in fact names of fractions, must play a prominent role and will play the role played by what is termed fractions in other approaches. For instance in the fractions as values paradigm the name of a fraction, rather than the fraction itself, is supposed to be equipped with a numerator and a denominator.

Rollnik argues at length that many approaches to fractions found in the literature and in existing teaching materials lead to mistakes, imprecision, and even contradictions, and he argues that fractions as a values is the better choice. Similar but less comprehensive criticism is formulated by Opmeer in [32]. The proposal of Rollnik depends, however, on the ability to provide a useful account of names of rational numbers.

Now unfortunately the very notion of a name is not so simple and its analysis has lead to intricate philosophical ramifications.⁵ For recent work on names see Gray [19]. Gray explains that a philosophical theory of names seeks to explain what a formalised logical theory of names in the tradition of analytical philosophy intends to avoid: the complications arising with the use of names in natural language.⁶ In the case of fractions, naming conventions seem to feature complications that are intrinsically linked to the use of natural language.⁷ Following Gray’s view in that matter the definition of a fraction that will be provided below using the fractions as terms paradigm, intentionally, or at least consciously, avoids giving an account of naming rational numbers. Indeed the definition that

³A thorough argument for the necessity of being able to give (an teach) a proper definition of fractions, following the fractions as values paradigm is given by Greenleaf in [20].

⁴Pot ([28]) lists several problems and ambiguities with the term fraction and he proposes to use the term ‘breuktal’ (fractional number) instead and to view that as a rational number. Given $r = p/q$ the numerator and denominator are taken to be to be the factors of p , resp. q that are obtained after simplification. This approach qualifies as a fractions as values paradigm.

⁵For instance: is “ $\frac{1}{2}$ is smaller than 1” a so-called bare use of the name $\frac{1}{2}$ or is it a predicative use? In [18] one finds the position that the distinction between bare use and predicative use is a context-sensitive one, thus opposing a famous view by Burge in [15], who made the suggestion that names are predicates. It is possible to view $\frac{1}{2}$ as a predicate on descriptions of numbers?

⁶To get a sense of these complications one may consider the following utterances: some Alfreds were film directors; Alfred is an Alfred; Alfred is driving the car today; after which Alfred an important prize has been named; a millennium ago there were no Alfreds; only Alfreds know what it means to be an Alfred.

⁷The informal language of fractions seems to invite the use of say $1/2$ as a value and as a name for that value at the same time, irrespective of the confusion that may arise from such ambiguous use of the term.

I will propose in Section 3.1 and the story of fractions based on that definition cannot be taken for a substitute of a theory of names for numbers.

1.2 Some difficulties with the notion of a fraction

Besides students of all ages being prone to making a range of characteristic mistakes when dealing with fractions, certain conceptual difficulties can be frequently observed in teaching methods and materials on fractions. The following list is a non-exhaustive survey of difficulties, including some observations due to Rollnik, which one may spot in a range of different presentations on fractions. After each assertion that specifies a viewpoint occurring “in practice”, I have given in brackets an explanation of why I consider that assertion problematic.⁸

1. One can only add fractions with the same denominator. (Fractions can always be added, the given restriction on denominators applies to adding by means of the so-called quasi-cardinality rule, see Paragraph 4.2.3 below, itself an instance of distribution of multiplication over addition.)
2. In order to compare the size of fractions they must be brought in the form of having the same denominator. (Having them in the form with equal nominators is just as useful for the purpose of comparison.)
3. In order to transform two fractions with denominators p and q respectively to the state of having the same denominator one must transform each to the state of having r as a denominator with r equal to the smallest common multiple of p and q . (In some cases one of the fractions or even both of them can better be simplified first.)
4. Fractions are stated to be rational numbers in combination with the claim that each fraction has a numerator and a denominator. (Rational numbers have no such attributes.)
5. Presentations will introduce fractions as pairs of integers written as, say $\frac{a}{b}$, and when fraction equivalence is explained as a relation on pairs another notation is used, say (a, b) instead of $\frac{a}{b}$. Presentations may then fail to notice that equivalence between fractions $\frac{a}{b}$ and $\frac{c}{d}$ is written $\frac{a}{b} = \frac{c}{d}$ instead of $\frac{a}{b} \equiv \frac{c}{d}$ or instead of $(a, b) \equiv (c, d)$. (Typically explanations of

⁸Stefan Rollnik has remarked on on this listing of issues that the first three of these items seem to be of a more general kind in the following sense: the word “must” and the phrase “can only” are often mistaken in elementary mathematical texts because there are different ways to proceed towards the intended objective from a particular intermediate state of an argument. If the use of “must”, or of “can only” is weakened to the mere formulation of an advice, the issues listed under items 1, 2, and 3 become non-specific for fractions. I agree with this observation, and it weakens the case that I am making. Two replies are possible: (a) even phrased as advices the mentioned items are problematic, (b) if a “must” or “can only” is taught in a particular case while an advice is meant, this state of affairs is covered by the requirements on teaching as listed in 5.2.3 below. That means that the teacher will need to explain at some predetermined moment that “must” or “can only” provide a simplified picture that is not valid in all circumstances.

the construction of rational numbers via equivalence classes of pairs fail to explain the relation between the pairs involved and the notion of a fraction. More specifically: are the pairs meant to formalize fractions somehow, or are they just unrelated to fractions, the ‘correspondence’ between (a, b) and $\frac{a}{b}$ being incidental rather than intentional?)

6. The suggestion is made that while for teaching purposes a naive story on fractions suffices, at an academic level (in German ‘Hochschulmathematik’) a rigorous approach to fractions will be (or can be) provided, an approach which is in fact based on the construction of rational numbers as equivalence classes of signed and unsigned pairs of natural numbers. (This view lacks support as there is no indication that mathematical textbooks provide a definition of fractions in addition to the construction of rational numbers.)
7. In many presentations of fractions, decimal notation for naturals and integers is presupposed and the notion of a fraction is introduced by means of examples involving decimal notation only. (It is more often than not left open whether numerator and denominator are to be viewed as numbers or as notations for numbers. This leads to obvious questions the answer of which is then left to the reader, such as whether or not $\frac{2+3}{7}$ and $\frac{5}{7}$ are the same fraction. The syntax of decimal notation being judged ‘complex’, teaching a rigorous position towards the distinction between syntax and semantics is avoided in the presence of decimal notation.)

Often one encounters the requirement on fractions that the denominator of a fraction may not be 0. As obvious and conventional as this restriction may appear, it mixes syntax and semantics in a non-obvious manner, for instance in order to ‘see’ that ‘the fraction’ $\frac{2+7}{1+((7-5)-3)}$ is wrong one needs to perform a valid calculation on its ‘denominator’, but not being a fraction it has no denominator. Perhaps the mentioned expression is a candidate fraction with a candidate denominator but such language is never used to the best of my knowledge. Sometimes the claim is made that the denominator of a fraction must not be 1.

Frequently the notation $p \div q$ (in German texts as well as in Dutch texts mostly written as $p : q$) is used but the status of that notation varies. Some authors claim that $p \div q = \frac{p}{q}$ serves as a defining equation for $- \div -$. One may consistently hold the converse as well. Some authors view both notations as synonyms, alternatives that can be exchanged within the same text, on the same page, and even in the same formula; yet working with a fraction written as $\frac{3+5}{5 \div 2}$ would be considered highly unusual.

1.3 Two more paradigms

I failed to find a convincing definition of a fraction as a mathematical concept. This leads me to the hypothesis that in spite of its abundant use as a technical

term, fraction is not a mathematical notion. Even in the case of fractions of integers, which constitutes by far the most ubiquitous case, and which throughout the world provides the foundation of teaching in arithmetic, several variations exist.⁹ There seems to be room for further research at this point: why is it the case that fraction just like proof, definition, theorem, and result, need not be given a rigorous definition in the presentation of arithmetic. In other words: what makes fraction as a notion different from say the following notions: integer, prime, rational numbers, real number, factor, field, metric space, or topological space, all notions for which giving rigorous definitions is standard practice. One easily finds two alternative paradigms on fractions: (i) viewing pairs rather than single values as the meaning of fractions, the so-called fractions as pairs paradigm, and (ii) taking fractions for syntactic expressions upon option of a distinction between syntax and semantics, what I will call the fractions as terms paradigm.

1.3.1 Fractions as pairs paradigm

Fractions as pairs is at first sight a fully viable approach which locates fractions as elements of a suitable mathematical domain. It is plausible to write a pair intended to denote a fraction as $\frac{p}{q}$ or interchangeably as p/q , where it is perfectly plausible to assume that q is nonzero. In this case p is referred to as the numerator of the fraction and q as its denominator.

A complication with the fractions as pairs paradigm arises if one asks the obvious question as to which operators are defined on fractions. It is plausible to assume that negation (additive inverse), multiplication, and division are defined on fractions by means of the following equations respectively:

$$-\frac{p}{q} = \frac{-p}{q}, \frac{p}{q} \cdot \frac{r}{s} = \frac{p \cdot r}{q \cdot s}, \frac{p}{q} / \frac{r}{s} = \frac{p \cdot s}{q \cdot r}.$$

A difficulty arises with addition, however. The most plausible defining equation for addition is the conditional axiom CFAR (conditional fraction addition rule):

$$q \neq 0 \wedge s \neq 0 \rightarrow \frac{p}{q} + \frac{r}{s} = \frac{p \cdot s + q \cdot r}{q \cdot s}.$$

In CFAR equality must now be understood as equality of pairs. With that reading, however, one finds $1/2 + 1/2 = 2/4$ and not $1/2 + 1/2 = 2/2$ whence it fails to meet an equally plausible requirement, the so-called quasi-cardinality rule (QCR).¹⁰ QCR asserts that

$$\frac{p}{q} + \frac{r}{q} = \frac{p+r}{q}.$$

It follows that either QCR is to be considered invalid in the case of fractions as pairs, or the definition of addition should not comply with CFAR, or some

⁹The presence of this variation goes well with the relativism as formulated in [35].

¹⁰QCR is a phrase ascribed to Griesel [21] in Padberg [27]. Using QCR one finds $1/2 + 1/2 = (1+1)/2 = 2/2$.

status difference between QCR and the CFAR of addition is assumed which allows one to read equality differently in both cases.

Now a ramification with three different options arises: (i) not to have addition as an operation on fractions, (ii) to define addition without regard to the quasi-cardinality rule, and (iii) following [3], to modify the rule for addition so that instead of the product of both denominators the resulting fraction has the smallest common multiple of both denominators as its denominator (see also Paragraph 2.5 below). Option (iii) is attractive except for the fact that it makes elementary arithmetic on fractions dependant on the presence of other operations on integers such as the greatest common divisor and integer division. Another problem with the fractions as pairs view is that one cannot simply refer to the following entities as fractions:

$$\frac{(\frac{1}{2})}{3}, \frac{1 + \frac{1}{2}}{3}, \frac{(\frac{1}{4})}{(\frac{3}{2})}$$

When considered as fractions, the rules for multiplication and division must be applied and the denominator of these fractions turns out to be 6, 2, and 3, respectively in spite of an appearance that suggests otherwise.¹¹

Summing up my assessment is that fractions as pairs is not the obvious paradigm of choice for fractions mainly because (i) it leaves open certain non-trivial design decisions, and (ii) it allows a rather unfortunate discrepancy between the visual extraction of components (numerator and denominator) and the mathematical definition of those components.

1.3.2 Fractions as terms paradigm

In Section 3.1 a definition of fractions is proposed based on the distinction between syntax and semantics. This distinction is made in the style of first-order logic. Fractions are viewed as a syntactic category, that is a set of terms or expressions, rather than as a semantic category. This constitutes the fractions as terms paradigm. There a complication, however, in that some key properties of fractions (in particular safety) can be defined relative to some particular semantic model only. It should be noticed, moreover, that most introductions to fractions do without any mention of a distinction between syntax and semantics.

The definition is given in the context of a meadow only (see Paragraph 2 below for some remarks on meadows). Working with meadows introduces a simplification in several ways: (i) by working with the syntax of fields expanded with function names for inverse and division an unambiguous and very simple notion of syntax becomes available, (ii) issues concerning division by zero are dealt with within the theory instead of being derived from an external view,¹²

¹¹These remarks are specific for a fractions as pairs view. With fractions as values in mind one may say that $(1 + 1/2)/3$ is a name of a fraction (i.e. rational number) with $1 + 1/2$ as a numerator. With fractions as terms in mind one may say that $(1 + 1/2)/3$ is a fraction with numerator $1 + 1/2$.

¹²A typical instance of an issue concerning division by zero that is simplified by working in a meadow runs as follows. Consider the rule $y \cdot v \neq 0 \rightarrow x/y + u/v = (x \cdot v + y \cdot u)/(y \cdot v)$.

(iii) with involutive meadows constituting a variety, equational logic becomes a useful tool.

If one accepts that, when confronted with the task to define fractions in a rigorous manner, the context of involutive meadows provides a simplification, it should be expected that in other contexts where meadows are unavailable, including the contexts available in primary school when teaching fractions begins, defining fractions is not straightforward. Nevertheless I consider it hardly plausible that in a different context defining fractions as a syntactic category will fail to be the most clarifying option, in spite of the fact that explicitly distinguishing syntax and semantics seems to go against long standing mathematical intuitions.

1.4 Requirements on a definition of fractions

Defining fractions is not an entirely open ended issue and some requirements must be met. Looking at the three approaches mentioned above: fractions as values, fractions as pairs, and fractions as terms, it is hard to see which joint set of requirements is met by the various definitions of a fraction arising in these three approaches.

An attempt to perform requirements engineering applied to the notion of a fraction leads me to the suggestion that the most basic requirements on its definition seem to be these:

- The concept of a fraction either (i) coincides with rational number, or (ii) it constitutes a form of representation thereof. Representations of rational numbers can take two forms: (ii-a) a logical form, that is a term or expression in a syntax with rational numbers as a model, or (ii-b) a mathematical form, say a pair or a triple made up from known mathematical objects. Alternatively, as option (iii), fractions may be defined as a mix of these three interpretations.
- A definition of fractions supports the development of a comprehensible ‘story on fractions’.
- A definition of fractions together with the development of theory about fractions based on that definition must provide a theoretical background for a conceivable curriculum on arithmetic that is workable at all levels of education.
- A definition of fractions together with its supporting theory is preferably independent of decimal notation. In other words, versions of such definitions and theories pertaining to binary notation, hexadecimal notation, or unary notation, ought to be available via simple and natural modifications.

This rule will be named CFAR below. Now justifying the the validity of CFAR is not an obvious matter. Its validity implies, or requires, the validity of its substitution instance $\Phi_0 \equiv 0 \cdot 0 \neq 0 \rightarrow 0/0 + 0/0 = (0/0 + 0/0)/(0 \cdot 0)$. That validity is most easily assessed if $0/0$ has a defined value. Unless that choice is made one either needs a logic of partial functions at this point, or a three-valued logic, or a short-circuit logic. Each argument for the validity of Φ_0 based on first order logic requires $0/0$ to have a value.

It is tempting to require of a definition of fractions and a story on fractions based on that definition that it supports existing course material, but that requirement runs counter to the observation that such material is often conceptually inconsistent, for instance if it incorporates one of the difficulties listed in Paragraph 1.2 above.

2 Technical preliminaries

An involutive meadow is a von Neumann regular ring (vNRR) expanded with an inverse operator (written x^{-1}) which assigns to an element x its so-called pseudo inverse.¹³

2.1 Preliminaries on meadows

For equational axioms for meadows see [12, 4, 25, 31] and for further theoretical information I refer to [5]. A field expanded with a multiplicative inverse that is made total by taking $0^{-1} = 0$ becomes an involutive meadow. Below meadow will by default be taken to refer to an involutive meadow. Involutive meadows share the property that for all x , $(x^{-1})^{-1} = 1$.¹⁴

Meadows can be presented in divisive notation as well, then division is used as the additional operator symbol instead of inverse, for a systematic comparison between both presentations of meadows see [6]. I will assume that both operations are available where division is defined in terms of inverse as $\frac{x}{y} = x \cdot y^{-1}$. A meadow is trivial if $0 = 1$, otherwise it is non-trivial. A cancellation meadow is a meadow that satisfies the general inverse law: $x \neq 0 \rightarrow x \cdot x^{-1} = 1$ (see also Paragraph 4.2 below). A cancellation meadow which is reduced by forgetting its inverse operator becomes a field. A meadow is minimal if it has no proper substructures. The meadow of rational numbers \mathbb{Q}_0 is a minimal meadow.

A numeral is a sum of a finite number of units. For numerals we use the conventional notation with $k \in \mathbb{N}$: $\underline{0} = 0, \underline{k+1} = \underline{k} + 1$. A polynomial term is a term in which neither division nor inverse occurs. Trivially all numerals are closed terms.

¹³The difference between a meadow and a vNRR can be appreciated as follows: meadows constitute a variety while vNRR's do not. In particular the class of meadow is closed under taking substructures while the class of VNRR is not.

¹⁴At least three other kinds of meadows can be distinguished: common meadows ([8]), wheels ([16, 17, 34], and transrationals ([29]). The characteristic feature of meadows at large is the presence of either inverse or division as a function symbol in the signature, in addition to the signature of a ring. This difference with conventional approaches has significant impact on equational reasoning which it supports quite well, and it fits the ubiquitous use of division. When working with meadows issues about division by zero and partiality can be studied with adequate precision.

2.2 The quasi-cardinality rule (QCR)

The following rule called QCR for quasi-cardinality rule plays a central role in stories on fractions. It was already mentioned above in Paragraph 3.

$$\frac{p}{q} + \frac{r}{q} = \frac{p+r}{q}.$$

QCR is provable from the axioms of meadows. However, if one assumes that division is defined on the basis of inverse by $x/y = x \cdot y^{-1}$ then QCR is merely a rephrasing of distribution of multiplication over addition, a fact that is comprised in CR. In the absence of inverse that viewpoint is less adequate because one needs some additional axiom, e.g. $p/q = (1/q) \cdot p$ in order to derive QCR from CR.

Below two additional versions of QCR will be distinguished, QCRfp which is specific for fractions as pairs, and QCRft which is specific for fractions as terms. Using this style of notation it is plausible to view QCR, assuming that it is considered an assertion about fractions rather than a mere assertion in the language of meadows, as an assertion which is specific for fractions as values. Following that convention the rule name QCR serves as a default for a more systematic rule name QCRfv which then is supposed to be specific for fractions as values.

The following conditional version CQCR of QCR will be used in Paragraph 2.5 below:

$$q \neq 0 \rightarrow \frac{p}{q} + \frac{r}{q} = \frac{p+r}{q}.$$

2.3 Fracvalues, fracpairs, and fracterms

In the three paradigms three different domains play a role, each leaving room for variation. It is helpful to have names for the roles of these domains available. In a fractions as values approach a domain of fracvalues contains these values, in a fractions as pairs approach a domain of fracpairs contains such pairs, and in a fractions as terms approach a domain of fracterms provides those terms. Unlike for fracvalues, for fracpairs and for fracterms a numerator and a denominator exist.¹⁵

It is plausible to use fracvalues, fracpairs, and fracterms at the same time, in which case there is no need for the term fraction, as a more precise language is already available. In such circumstances it is also plausible that fracpairs are viewed as an abstraction of fracterms and in turn fracvalues as understood as an abstraction of fracpairs.

In the setting of a meadow fracvalues are simply elements of the meadow and fracpairs are pairs $\frac{p}{q}$ with p and q elements of the meadow. Fracterms for meadows will be discussed in Section 3.1 below.

¹⁵In [9] the term *fracpair* is introduced in an approach which takes a fraction for an equivalence class of a pair of integers, or more generally for a pair of elements of a reduced ring. That use of fracpairs allows a denominator with value zero. The same paper uses fracterms but it makes no use of fracvalues, a term that is introduced in the present paper.

2.4 Fracvalue equality, fracpair equality, and fracterm equality

It is plausible to insist that equality of fracvalues p and q is written as $p = q$. Then fracpair equality of fracpairs α and β can be written $\alpha \cong \beta$ and fracterm equality of fractures P and Q can be written as $P \equiv Q$. Because each fracterm can be considered a fracpair and each fracpair can be considered a fracvalue, it is meaningful to use $P = Q$ and $P \cong Q$ for fracterms as well. Then for all fracterms P and Q , $P \equiv Q \implies P \cong Q$ and $P \cong Q \implies P = Q$.

The quasi-cardinality rule QCR will now appear in a second version as well: QCRfp: $\frac{p}{q} + \frac{r}{q} \cong \frac{p+r}{q}$.¹⁶

With this notation at hand it is possible to write with more precision about the complications concerning fracpairs which were mentioned above in Paragraph 3.

A defining conditional equation for addition of fracpairs might be considered as follows: $y \neq 0 \wedge v \neq 0 \rightarrow x/y + u/v \cong (x \cdot v + y \cdot u)/(y \cdot v)$. That (candidate) defining equation fails to comply with QCRfp. Counterexamples to QCRfp exist with a non-zero denominator, e.g. consider $1/2 + 1/2$. Given the importance of QCR, something else must be sought for the addition of fractions as pairs.

2.5 Fracpairs over integers

It is plausible to have integers rather than the meadow of rational numbers as the domain from which to compose fracpairs. In that case the discrepancy with QCR can be settled by means of a definition based on CFARfp, the conditional fraction addition rule for fractions as pairs:

$$q \neq 0 \wedge s \neq 0 \rightarrow \frac{p}{q} + \frac{r}{s} \cong \frac{(p \cdot s + q \cdot r) \setminus \gcd(q, s)}{(q \cdot s) \setminus \gcd(q, s)}.$$

Here “ \setminus ” represents integer division and gcd produces the greatest common divisor.¹⁷

The conditional version CQCRfp of QCRfp (see Paragraph 2.4 above) reads as follows:

$$q \neq 0 \rightarrow \frac{p}{q} + \frac{r}{q} \cong \frac{p+r}{q}.$$

Each closed instance of CQCRfp follows from CFARfp. However, as a definition of addition on fracpairs needs to take the anomalous cases with a denominator equal to zero into account CFARfp does not suffice for a definition of addition on fracpairs.

Now QCRfp (that is CQCRfp without its condition) may be satisfied as well by completing the definition of addition for fracpairs with the following three

¹⁶A third version of QCR, QCRft ($\frac{p}{q} + \frac{r}{q} \equiv \frac{p+r}{q}$) may be contemplated but it will trivial as it will always be false.

¹⁷When working out the details of this definition one finds that integer division needs to take 0 as a second argument into account and one is led into contemplating Euclidean meadows involving integer division, remainders, and the assumption that $n \setminus 0 = 0$.

(conditional) axioms.

$$q \neq 0 \rightarrow \frac{p}{q} + \frac{r}{0} \cong \frac{p}{q}, q \neq 0 \rightarrow \frac{p}{0} + \frac{r}{q} \cong \frac{r}{q},$$

and

$$\frac{p}{0} + \frac{r}{0} \cong \frac{p+r}{0}.$$

In the presence of conditional axioms that forget the numerator of a fraction (in an addition) with zero denominator, replacing the last equation by $\frac{p}{0} + \frac{r}{0} \cong \frac{0}{0}$ seems to be more natural, however, as this equation forgets both numerators. This advantage may be acknowledged in spite of the fact that it is at cost of the unconditional QCRfp, which suggests that in the case of fractions as pairs imposing QCRfp is more plausible than imposing QCRft.

2.6 Paradigm terminology once more

The three paradigms can now be informally understood as follows, one chooses an option from: values, pairs, or terms, then one chooses a corresponding domain for fracvalues, fracpairs, or fracterms and one develops a story in which (a) fractions are understood as fracvalues, as fracpairs, or as fracterms, according to the option chosen, and (b) one writes all equalities with $=$ and expects the reader to perform type inference when needed.

3 Fractions and fractional numbers

In the case of meadows a fractions as terms view is quite plausible. The details of that approach will now be worked out without use of the unusual jargon of fracvalues, fracpairs, and fracterms. In some case use is made of \cong and \equiv .

A fractions as terms view presupposes a distinction between syntax and semantics. Moreover, it leads to a distinction between open fractions and closed fractions and in addition it calls attention to the question how to refer to the elements of a meadow denoted by closed fractions. In the case that a meadow is used as a structure for numbers it is plausible to refer to such elements as fractional numbers. These intuitions are made precise below.

3.1 Defining fractions

Definition 1. *An arithmetical term (alternatively arithmetical expression) is a term over the signature of meadows (using divisive notation, that is not using inverse).*

Definition 2. *A closed arithmetical term is an arithmetical term which contains no variables.*

Definition 3. *A fraction is an arithmetical term with division as its leading function symbol.*

Clearly each fraction has the form $\frac{p}{q}$ for some terms p and q . Fractions can be open and closed. Each arithmetical term is either a fraction or a non-fraction. A term that has no fractions as subterms is (provably equal to) a polynomial term.

Definition 4. *A closed fraction is a fraction which also is a closed arithmetical term.*

Definition 5. *For a fraction $\frac{p}{q}$ the term p is called its numerator and the term q is called its denominator.*

Definition 6. *A fraction is called flat if its numerator and its denominator are both polynomial terms.*

Definition 7. *A fraction is composed if it is not flat.*

A comprehensive story on fractions needs more aspects than the definitions given above. Formulating additional properties of fractions require that one assumes the existence of a fixed non-trivial meadow relative to which classes of fractions are to be defined. We assume that A is a non-trivial meadow (i.e. $0 \neq 1$).

Definition 8. *A fraction is a common fraction relative to A , if its denominator is not equal to 0 in A . A fraction is uncommon if it is not common.*

Definition 9. *A fraction is safe w.r.t. A if it is a common fraction and if none of its proper subterms are uncommon fractions.*

Definition 10. *An arithmetical term is safe w.r.t. A if either it is a safe fraction or none of its subterms are uncommon fractions.*

Some subsets of the class of all closed fractions have been given names in the literature.

Definition 11. *Given a non-trivial meadow A , the following classes of terms are considered relative to A :*

1. *A simple fraction is a common fraction of the form $\frac{p}{q}$ with both p and q a numeral.*
2. *Two simple fractions $\frac{p}{q}$ and $\frac{r}{s}$ are equivalent in A if $A \models p \cdot s = q \cdot r$.*
3. *A unit fraction is a common fraction of the form $\frac{1}{q}$.*
4. *A simple fraction is in simplified form if it is of the form $\frac{p}{q}$ where for some $k, l \in \mathbb{N}$ with k and l relatively prime, $p \equiv \underline{k}$ and $q \equiv \underline{l}$.*
5. *A proper fraction is a simple fraction of the form $\frac{p}{q}$ with for some $k, l \in \mathbb{N}$ with $k < l$, $p \equiv \underline{k}$ and $q \equiv \underline{l}$.*
6. *An improper fraction is a simple fraction of the form $\frac{p}{q}$ with for some $k, l \in \mathbb{N}$ with $k \geq l$, $p \equiv \underline{k}$ and $q \equiv \underline{l}$.*

7. *German: Scheinbruch. A Scheinbruch is a simple fraction of which the numerator is a multiple of the denominator.*
8. *Mixed fractions are an extension of fractions rather than a particular kind of fractions. Thus mixed fractions are not fractions according to Definition 3.*

A mixed fraction is an expression of the form $\underline{n}\frac{p}{q}$ or of the form $-\underline{n}\frac{p}{q}$ with n a positive integer and $\frac{p}{q}$ a positive proper fraction. $\underline{n}\frac{p}{q}$ is an abbreviation of $\underline{n} + \frac{p}{q}$ and $-\underline{n}\frac{p}{q}$ abbreviates $-(\underline{n} + \frac{p}{q})$.

3.2 Fractional numbers in a meadow

A closed fraction denotes a unique value in a given meadow. Assuming that a meadow is used as a structure for numbers a closed fraction denotes a number. Now the identity $x = \frac{x}{1}$ holds in each meadow and therefore is each element of a minimal meadow the interpretation of a closed fraction. This observation motivates the following definition.

Definition 12. *Given a meadow A , a value in A that is denoted by a closed flat fraction is called a fractional number.*

For a closed flat fraction P the fractional number denoted by it is written $A \models P$ or preferably $[P]_A$.¹⁸ In the meadow of rational numbers \mathbb{Q}_0 all elements (numbers) are fractional numbers.

This rigorous notation ($[P]_A$) for the interpretation of fractions as fractional numbers being somewhat heavy, one is tempted to delete the subscript A if it is known throughout a context, and if possible without creating confusion one is tempted to write P instead of $[P]$ as a further simplification.

In elementary arithmetic it is customary to have $A = \mathbb{Q}_0$ in mind, so that one tends to write P both for the term proper and for the corresponding fractional number $[P]_{\mathbb{Q}_0}$. A particular occurrence of the use of this shorthand may inadvertently create the impression that the transition to a fraction as value view is implicitly occurring, that is a view where a distinction between fractions and fractional numbers is intentionally not made in a systematic manner. Nevertheless if sufficient explanation is provided it must be possible to write P where $[P]_A$ is meant and to expect a reader to perform the required type inference in order to allow the unambiguous reading of a text.

3.3 Closure properties of the fractional number set

In \mathbb{Q}_0 all numbers are fractional numbers and therefore the set of fractional numbers is closed under multiplication, negation, inverse, as well as addition. The general case for an arbitrary meadow is less straightforward, however.

¹⁸In German fraction may be translated as *Bruch*. For ‘Bruch’ P , $[P]_A$ is called its *Bruchzahl*.

In each meadow the following three equations hold: $\frac{x}{y} \cdot \frac{u}{v} = \frac{x \cdot u}{y \cdot v}$, $(\frac{x}{y})^{-1} = \frac{y}{x}$, and $-\frac{x}{y} = \frac{-x}{y}$. It follows from these identities that fractional numbers are closed under multiplication, division, and negation.

The more interesting case is addition. It can be shown that in the initial meadow (see [14]) the set of fractional numbers is not closed under addition. As a consequence there is no general equation that expresses the sum of two fractional numbers in a meadow, in other words the set of fractional numbers of a meadow need not constitute a meadow.

This leaves us with the question what in general can be said about the sum of two fractional numbers $[P]_A$ and $[Q]_A$ in a meadow A .

4 Addition of fractions

The most plausible candidate for a rule for addition two fractional numbers is the well-known identity $\frac{x}{y} + \frac{u}{v} = \frac{x \cdot v + y \cdot u}{y \cdot v}$. This rule will be studied in more detail below and it will be adapted to a conditional form. The main virtue of this identity is that it allows to transform all closed expressions including all fractions to flat fraction form.

4.1 Unconditional fraction addition rule

The (unconditional) axiom FAR (fraction addition rule) is as follows:

$$\frac{x}{y} + \frac{u}{v} = \frac{x \cdot v + y \cdot u}{y \cdot v}.$$

FAR is not valid in a nontrivial involutive meadow, because it implies $\frac{1}{1} + \frac{1}{0} = 0$ while such a meadow satisfies: $\frac{1}{1} + \frac{1}{0} = \frac{1}{1} + 0 = 1$. In the presence of FAR, $\frac{1}{0}$ behaves like a sink for addition and for multiplication:

$$u + \frac{1}{0} = \frac{u}{1} + \frac{1}{0} = \frac{0 \cdot u + 1 \cdot 1}{1 \cdot 0} = \frac{1}{0},$$

and

$$\frac{1}{0} \cdot u = \frac{u}{0} = \frac{1 + (u - 1)}{0} = \frac{1}{0} + \frac{u - 1}{0} = \frac{1}{0}.$$

It follows that adopting FAR brings one unavoidably in the setting of the so-called common meadows that have been defined and studied in [8]. When working in involutive meadows, FAR is too strong and it must be weakened which can be done introducing a condition.

4.2 Conditional fraction addition rule

The conditional fraction addition rule (CFAR) was introduced in Paragraph 3 as follows:

$$y \neq 0 \wedge v \neq 0 \rightarrow \frac{x}{y} + \frac{u}{v} = \frac{x \cdot v + y \cdot u}{y \cdot v}.$$

In the context of involutive meadows CFAR follows immediately from the so-called generalized inverse law (see [5]) $x \neq 0 \rightarrow \frac{x}{x} = 1$. Indeed $\frac{x}{y} + \frac{u}{v} = \frac{x}{y} \cdot \frac{v}{v} + \frac{y}{y} \cdot \frac{u}{v} = \frac{x \cdot v}{y \cdot v} + \frac{y \cdot u}{y \cdot v} = \frac{1}{y \cdot v} \cdot (x \cdot v) + \frac{1}{y \cdot v} \cdot (y \cdot u) = \frac{1}{y \cdot v} \cdot (x \cdot v + y \cdot u) = \frac{x \cdot v + y \cdot u}{y \cdot v}$.

In fact in the presence of the axioms Md of (involutive) meadows $x \neq 0 \rightarrow \frac{x}{x} = 1$ follows from CFAR. To see this first notice that $1 = 0$ implies $\frac{x}{x} = 1$ and thus one may assume $1 \neq 0$. Now notice that given $x \neq 0$ from CFAR one finds $\frac{1}{1} + \frac{x}{x} = \frac{1 \cdot x + 1 \cdot x}{1 \cdot x} = \frac{x + x}{x} = \frac{x}{x} + \frac{x}{x}$. Then subtracting $\frac{x}{x}$ from both sides yields $\frac{1}{1} = \frac{x}{x}$ which is the conclusion of $x \neq 0 \rightarrow \frac{x}{x} = 1$ had to be shown.

Following [5] a meadow satisfying $x \neq 0 \rightarrow \frac{x}{x} = 1$ is called a cancellation meadow. It follows that precisely in the cancellation meadows CFAR is valid. Moreover, trivially in a cancellation meadow the fractional numbers are closed under addition because a fraction with a denominator equal to 0 vanishes so that $\frac{x}{y} + \frac{z}{0} = \frac{x}{y}$.

4.2.1 A definition of addition using fracterm equality

As a conditional defining equation for fraction addition CFAR should be read as follows (CFARft):

$$y \neq 0 \wedge v \neq 0 \rightarrow \frac{x}{y} + \frac{u}{v} \equiv \frac{x \cdot v + y \cdot u}{y \cdot v}.$$

Assuming this defining equation as a requirement on the addition of fractions (as terms) is it not possible to require in addition that QCRft is valid, with QCRft the following identity:

$$\frac{x}{y} + \frac{u}{y} \equiv \frac{x + u}{y}.$$

Indeed, even stronger, for all closed terms t, r and s , $\frac{t}{r} + \frac{s}{r} \not\equiv \frac{t+s}{r}$.

4.2.2 Calculating closed fractions

For each closed arithmetical term t there is a simplified closed flat fraction r such that:

$$\text{Md} + \text{CFAR} + \{\underline{1} \neq 0, \dots, \underline{k+1} \neq 0\} \vdash t = r.$$

Moreover if t is safe then r can be chosen safe as well.

This fact has a straightforward proof with induction on the structure of closed arithmetical terms.¹⁹

¹⁹Finding r in simplified form given t is a typical exercise that may occur in teaching on fractions. In conventional education t and r are written in decimal notation. Decimal notation constitutes a significant theoretical overhead for meadow theory, and it is not dealt with in detail in this paper for that reason. In [10] the reader may find a survey of datatype specifications involving binary and decimal notation. These specifications, viewed as equational abstract data type specifications can be extended to the case of rational numbers, though not without compromising either confluence or termination or both. Moreover, in school mathematics it is usually taken for granted that repeated additions can be written without brackets. In [11] the notion of a poly-infix operator has been proposed which allows omitting brackets while not assuming (familiarity with) the concept of associativity as a foundation for that convention.

4.2.3 Working with a modified set of rules

Calculating fractions, that is finding a simplified flat fraction provably equal to a given closed fraction under the hypothesis that suitable numerals are nonzero, can be done without making use of the three equational axioms for divisive meadows regarding division (that is without $1/(1/x) = x$, $(x \cdot x)/x = x$, and $x/y = x \cdot 1/y$) by making use of equations that may be considered more familiar.

The axioms CR for commutative rings hold in all meadows and it is plausible to look for extensions of CR that allow the calculation of fractions. Now calculation of closed fractions can also be obtained with CR + QCR + DBZ + DIV1 + DIV2+ FEQ with DBZ, DIV1,2 and FEQ as follows. DBZ is the equation

$$\frac{x}{0} = \frac{0}{1}$$

DIV1 is the equation

$$\frac{(\frac{x}{y})}{z} = \frac{x}{y \cdot z}$$

and DIV2 is the equation

$$\frac{x}{(\frac{y}{z})} = \frac{x \cdot z \cdot z}{y \cdot z},$$

while FEQ (for fraction equivalence) is the scheme: for $k \in \mathbb{N}$, $k > 0$

$$\underline{k} \neq 0 \rightarrow \frac{x}{y} = \frac{x \cdot \underline{k}}{y \cdot \underline{k}}.$$

It is worth noticing that due to the assumption that one is working in the setting of an involutive meadow the condition that y and z are nonzero is not needed in DIV1,2.²⁰

The assumption $\underline{k} \neq 0$ is taken for granted in school arithmetic, where characteristic zero is an implicit assumption, but unavoidably at some point the assumption that one is not working in a finite field (meadow) enters the story.

4.3 Division safe calculation

One may object to the calculation $\frac{1}{1} + \frac{1}{0} = \frac{1}{1}$ because it identifies an unsafe term with a safe flat fraction. Such transformations may create risks when meadow based calculations are applied in practice. Unsafe expressions occurring in an application of working with rational numbers represented by means of fractions as outlined above, constitute an indication of the presence of either a design fault or of a modelling fault or of the simultaneous presence of both. In other words, by taking involutive meadows as a point of departure it becomes a design rule on the application of rational numbers that the application allows for division safe

²⁰DIV2 is somewhat more involved then one might expect. That has been done in order to ensure its safety, that is, used as a rewrite rule from left to right only the rule turns safe fracterms into safe fracterms and unsafe fractures into unsafe fracterms.

calculation of values (fractional numbers). Providing an ad hoc efficient strategy for division safe calculation may be part of a specific modelling effort.²¹

Calculation may be protected against the risk of calculating away unsafe expressions by restricting the rule by disallowing the DBZ from CR + QCR + DBZ + DIV1 + DIV2 + FEQ. That is when working in CR + QCR + DIV1 + DIV2 + FEQ each safe closed expression t can be calculated as to find the form of a closed flat fraction r in such a manner that for an arbitrary (possibly unsafe) input expression t' a expression r' which might be obtained by applying the same algorithm is safe if and only if the expression t' is safe. Both false positives and false negatives are avoided but the possibility of divergence on an unsafe input term is left open. As a byproduct this calculation produces a listing of conditions of the form $\underline{k} \neq 0$ which are sufficient for the equational derivation of $t = r$ from CR+QCR+DIV1+DIV2 + FEQ.

For an alternative approach to division safe calculation in meadows I refer to [13].

4.3.1 Common meadows as an alternative

Alternatively one may turn to the common meadow \mathbb{Q}_a of rational numbers as a structure in which to define fractions. In a common meadow the inverse of 0 is defined to be an additional value a which serves as an error element which propagates through all functions. A common meadow is not involutive because $a^{-1} = a$ instead of $a^{-1} = (0^{-1})^{-1} = 0$. Safety for an expression is now modeled as being different from a .

In a common meadow FAR holds unconditionally, which may be viewed as a simplification when compared with the case for involutive meadows.

4.3.2 A comparison

Division safe calculation in the involutive meadow \mathbb{Q}_0 is incomplete w.r.t. equality in that structure. A justification, rather than an explanation, of working with the restricted set of equational axioms CR + QCR + DIV1+ DIV2 + FEQ is given by the theory of fractions for involutive meadows. This discrepancy constitutes a mismatch which I consider to be the major disadvantage of the theory of fractions that has been put forward above.

Calculation in common meadows is by definition division safe. But the complexity of the axioms for common meadows as given in [8], which are significantly less elegant than those for involutive meadows, constitutes a definite disadvantage in comparison with the involutive case.

I have a preference for involutive meadows over common meadows regarding this dilemma. The advantage of simplicity of equational axioms for involutive meadows outweighs the built in protection, as offered by common meadows,

²¹Indeed no meadow based application involving rational numbers can be properly designed without taking these issues into consideration. This may be considered an advantage, in terms of providing additional design rules, as well as a disadvantage, in terms of not getting division safe calculation for free.

against ignoring non-safety of expressions during a calculation, even if this preference implies that in technical applications restricted rule sets such as CR + QCR + DIV1 + DIV2 + FEQ must be used.

There are many options for such rule sets and undeniably the design of such restricted rule sets introduces issues that are very close to the problem of finding an equational specification of the common meadow \mathbb{Q}_a , and of finding equations that hold in a suitable larger class of algebras which merits being called the class of common meadows.

5 Mixed paradigms, paradigm choice, and baseline selection

The problem of understanding fractions is far more complex than a mere distinction in three paradigms suggests. I will refer to the three paradigms as the central paradigms on fractions. Each central paradigm offers a possible view on the topic, but in practice people use various combinations and mixtures of the central paradigms.

5.1 Combining paradigms

Starting out from the central paradigms on fractions at least four different forms of paradigm combination may be distinguished.

Paradigm integration. One may have two or three central paradigms in mind and work with different sorts for fracterms, fracpairs, and fracvalues, while using the same notation for the equality over of each of these. Then disambiguation of a text requires type inference and each fracterm may be thought of as being implicitly labeled with a type indication: $(\frac{2}{3})_{ft}$ casts $\frac{2}{3}$ as a fracterm, $(\frac{2}{3})_{fp}$ as a fracpair and $(\frac{2}{3})_{fv}$ specifies it as a fracvalue. It is plausible to use $(\frac{2}{3})_{fv}$ as a default typing of $\frac{2}{3}$.

As an example assume that $\text{num}(-)$ represents a function that extracts the numerator from a fraction and consider the following chain of equalities: $1 = \text{num}(\frac{1}{2}) = \text{num}(\frac{2}{4}) = 2$. Obviously something is wrong with this alleged proof that $1 = 2$, the question is to explain what is wrong in detail. That explanation may work as follows: one assumes that expressions like $\frac{1}{2}$ have two possible types: fracpair and fracvalue. Lacking explicit type information the various equations must be enriched with type information in order to allow an unambiguous determination of meaning. There are two options: $1 = \text{num}((\frac{1}{2})_{fv}) = \text{num}((\frac{2}{4})_{fv}) = 2$ which fails because $\text{num}(-)$ requires its argument to be a fracpair, and $1 = \text{num}((\frac{1}{2})_{fp}) = \text{num}((\frac{2}{4})_{fp}) = 2$ which fails because there is no way in which $\text{num}((\frac{1}{2})_{fp}) = \text{num}((\frac{2}{4})_{fp})$ can be derived as $(\frac{1}{2})_{ft} \neq (\frac{2}{4})_{ft}$.

Spontaneous repeated paradigm alternation. An option is to view fractions as values and fractions as pairs not as competing views but as com-

plementary views that alternate in time in the mind of a person contemplating fractions. Switching back and forth between a fraction as a value view and a fraction as a pair view has been described in [3] where it is compared to the human perception of the Necker cube. As in the case of the Necker cube such paradigm switches may not be under control of the conscious mind.

Progressive paradigm development. Adopting a fractions as terms paradigm requires acceptance of a distinction of syntax and semantics. That adoption brings with it a cost that may be considered too high by someone insisting on keeping logic at a distance from mathematics. Now fractions as pairs may be considered a viable alternative to fractions as terms which avoids recourse to the notion of syntax, in spite of the disadvantages of that view which have been put forward in Paragraph 3.

This argument in favour of fractions as pairs can be turned around: the price of the fractions as terms paradigm is to take first order logic on board, or at least its equational fragment. That price may be considered high and it may only be considered justified to pay that price on the basis of substantial dissatisfaction with the fractions as pairs paradigm. In other words, fully understanding the rationale of the fractions as terms paradigm requires an awareness of enduring dissatisfaction concerning the fractions as pairs alternative.

One may therefore wish to understand fractions as values, fractions as pairs, and fractions as terms as three successive stages of theory design about fractions each of which finds part of its rationale in the conscious and deliberate rejection of its predecessor.

Context dependent paradigm selection. Finally rather than looking at fraction theory design as a process that stabilises in the mind of an individual at the maturity level of the fractions as terms paradigm, one may imagine that each confrontation with fractions invites a person to carry out this three stage theory design and revision process in the context of that particular confrontation with fractions and to use the first paradigm (in the ordering just mentioned) that fits the context irrespective of possible deficiencies of that paradigm in other contexts.

Thus one may imagine an ongoing repetition within the mind of users of the concept of fraction of the theory design and redesign cycle allowing to work with the simplest option that works in a specific context. For instance in a case where numerators and denominators are not used the fractions as values paradigm may be satisfactory, and in a context where there components are of importance but only flat fractions occur one may be satisfied with the fractions as pairs paradigm. Instead of choosing, motivating, and subsequently proposing a best choice from these three paradigms regarding the understanding of fractions, the operational readiness for the indefinite maintenance of a dynamic mechanism of context dependent paradigm selection emerges as a plausible conceptual option.

5.2 Paradigm choice and teaching fractions

As Albert Visser recently mentioned to me (September 2015, in Dutch, here rephrased in my own wording):

there are just two issues with fractions and the teaching of elementary arithmetic: (a) why and how does it matter for educational work to have a correct theory of fractions, and (b) what amounts correctness of theory to in this particular case.

Paradigm choice allows degrees of freedom concerning (a) as well as (b). Except for the previous Paragraph this paper contributes to question (b) rather than to question (a). Concerning (a) I have the following remarks.

No valid statement about educational usage of a theory in practice can be reliably made without performing empirical research. Therefore, in the absence of an empirical underpinning, in the case of fraction theory the intended link with educational practice must be formulated differently. I will use the language of conjectural abilities following the line of my work on outsourcing in [2]. The suggestion is made in [2] that a new theory (such as fraction theory), or a contribution to or extension of a theory is preferably complemented with a survey of conjectural abilities which plausibly result from adoption of the new theory. In the case of fraction theory the question at hand is: which conjectural abilities in an educational setting may follow from accepting fraction theory as outlined above?

I hope, and conjecture, that teaching staff who is aware of fraction theory can improve its ability to present a complete and consistent story on fractions which permits students to their advantage to engage in valid and reliable reasoning about fractions and numbers. This ability may be fruitful specifically in connection with students who are considered less gifted or talented concerning arithmetic in particular and mathematics in general. Whereas students for whom arithmetic and mathematics poses no problems will not need the conscious use of reliable and rigorous reasoning patterns, that may not hold for other students for whom sustained contemplation of the issues at hand may constitute vital prerequisite for the successful acquisition of calculational skills.

5.2.1 Paradigm choice and fraction components

One single issue stands out in terms of the relation between teaching and paradigm selection for fractions. If one opts for fractions as values then fractions don't have a numerator and a denominator. Stated differently, whenever each fraction is assumed to be composed of a numerator and a denominator which can be extracted from it, then another paradigm than fractions as values must underlie the story on fractions at hand. Also a distinction between simplified fractions and fractions that admit further simplification cannot be made in a pure fractions as values approach. Therefore insisting that fractions can be decomposed into different components limits the freedom of paradigm choice.

Similarly a fractions as pairs approach, which allows for numerators and denominators, and which provides a sufficiently sharp picture to distinguish

simplified fractions from non-simplified fractions, fails to support differentiating simple fractions from complex fractions.

5.2.2 Paradigm choice and theoretical baseline selection

Choosing (involutive) meadows as a theoretical basis that underlies one's fraction theory reaches further than paradigm choice about fractions. Each paradigm, and each combination or mixture of paradigms can be based on different underlying explanations among which the theory of meadows that relies on abstract datatypes, universal algebra, and equational logic, constitutes merely an option. I will refer to such a choice as a baseline theory, and baseline theories for fraction theory are not unique. It has been established in this paper, at least to my own satisfaction, that meadows provide an adequate baseline for fraction theory. Speaking of baselines in the context of fraction theory generates the following two questions at least: (a) why making use of a baseline at all, and (b) is there any reason to prefer an approach using meadows over other approaches to baseline theories.

5.2.3 Meadows as a theoretical baseline for fraction theory

Supposing someone adopts the fractions as terms paradigm and in addition that person is inclined to use involutive meadows as a semantic framework for the resulting story on fractions. Now assume that (s)he is about to design a teaching method one elementary arithmetic and is confronted with the following question: must meadows be taught before fractions (as terms). A positive answer to that question instantly renders the envisaged educational design project a mission impossible because it is highly unlikely that a student can be successfully told about meadows or before having been exposed to fractions in a simpler manner. A negative answer to the stated question seems unavoidable.

This negative answer suggests the subsequent question how in principle it can be the case that a meadow based background theory for fractions, or any comparable approach based on different mathematical structures, can matter for the development of such teaching methods. I hold that this second question allows a moderately positive answer. The idea of meadows constituting the background theory for the development of an educational path towards a story on fractions works thus.

1. The learning trajectory of a student may very well pass through stages where the student temporarily holds an inconsistent view of a topic. This applies in particular if obtaining a consistent view of a topic requires a theoretical complexity which can be better appreciated after the student has been made aware of the problem it is supposed to solve.
2. As a consequence of the previous observation it is also plausible that a teach feels the urge to assert 'facts' which may not remain unchallenged on the long run.

3. When teaching, a teacher is permitted to make assertions that (s)he considers invalid against the background theory on which the teaching is based.
4. However, when an invalid assertion is made by a teacher (or in teaching material used by the teacher) in order to help the student(s) reaching some stage of cognitive development, the teacher must take care of the following rules of engagement:
 - (a) be aware of that fact (the invalidity of the assertion against the preferred background theory),
 - (b) have a consistent story in mind based on the background theory,
 - (c) make sure that the presence of an inconsistency does not stand in the way of the development of useful reasoning patterns that weaker students may need to operate consciously and deliberately with the conceptual ingredients at hand,²²
 - (d) have a strategy in mind concerning how and when to inform the students about the ‘problem’ and how to install a valid theory in their minds by issuing a disclaimer and a subsequent adaptation of the assertion that was issued for temporary didactical reasons in spite of its known invalidity,
 - (e) be aware that the background theory at hand is not proven by its ability to help out with the conceptual problem at hand, other background theories might also provide that service, but some definite story is needed.
5. A teacher should have and effectuate a plan for providing a consistent story at the end. This plan is based on a consistent story (in our case on fractions). Such a story needs to be explicit about the underlying paradigm and the underlying semantics.

²²The seemingly obvious slogan that one can only explain something if one understands it oneself may stand in the way of appreciating this point. A talented student (and such was likely the condition as a student of the teacher him- or herself) may not need to be pointed out a clear statement on whether or not 1 is a fraction and may not need to be informed in general terms about an answer to the question which numbers are fractions after it has been asserted that fractions are numbers. Some students, however, upon having been told during a fractions as values based course that fractions are numbers, may get lost in the question which numbers are not fractions and why. A teacher who explains that all primes greater than 2 are odd is likely to explain that 9 is not a prime and that therefore not all odd numbers greater than 2 are primes. It should not be rejected beforehand that a student may expect and look for comparable clarity about fractions as about primes. While talented students may sense the magnitude of number theory and its dependence on clarity about primes in contrast with the auxiliary and flexible nature of the concept of fractions, such intuitions may not come easily to the less talented student.

I hold that a teacher who casually states that ‘fractions are just numbers, and there is no need for a more sophisticated viewpoint’, must be able to produce an instantaneous and stable answer to the question as to whether or not 1 is a fraction. If the teacher would answer that both options are possible or anything of that sort, the proof that a more sophisticated view is in fact needed in spite of the teacher’s assurance to the contrary, has been delivered on the spot so to say.

A possible application of these rules of engagement works thus in the case of fractions: if a teacher has chosen fractions as values is as the preferred paradigm, and if at some stage is stated by the teacher that fractions have a numerator and a denominator, it is required that at some later stage the students are told that in fact fractions have no numerator and denominator but that names for fractions (or any other terminology suggested by an available presentation of the chosen paradigm) have these components instead. It is also required that the timing of the later stage is known in advance and that it is made sure that this phase is not skipped.

Currently I think that fractions as terms based on involutive meadows constitutes the simplest choice for a background theory of fractions. There are many alternatives, however. A prominent alternative is to view division as a partial function. Doing so brings logics of partial functions into play which is a significantly more difficult subject than involutive meadows. Working with involutive meadows allows a simplification of the questions about division by zero by making use of an ‘overspecification’. Indeed there is no necessity for the idea that $1/0 = 0$, it merely finishes a story on numbers in a way from which a consistent story on fractions can be easily designed.

6 Concluding remarks

Requirements on a definition of fractions have been captured. A simple account of fractions has been given in the specific context of involutive meadows. The account follows the fractions as terms paradigm. The account meets the requirements formulated for a definition of fractions. Fraction theory viewed more broadly may emerge in time as a combination of context specific fraction theories. Fraction theory at large is not committed to the fractions as terms paradigm, and it may also include approaches based on paradigms that were not covered in the introduction.²³

This work may eventually find an application in the development of teaching material on fractions and arithmetic. Perhaps different strategies can be based on different paradigms in the storytelling on fractions. In fact I hope that meticulous definitions of fractions and related notions such as provided in this paper can be of help for developing teaching methods that are useful for students with a relatively weak talent for arithmetic. However, if making use of such variation in classroom practice leads to working with inhomogeneous groups other disadvantages related to lack of teacher attention, or to unevenly distributed teacher attention, must be taken into account (see [26]).

This work is trivial from a mathematical perspective, but nevertheless it took me much time and effort to come to believe the assumption underlying this work, namely that involutive meadows and storytelling on fractions are

²³The operator view of fractions as used in [1] is an example of an approach not covered in this paper. I left it out because I consider the operator concept to involve too much of a detour if the primary objective is defining fractions.

connected in a substantially meaningful manner. I am now confident that such a belief can be based on the arguments that have been put forward in this paper.

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